

The extended conformal theory of Luttinger systems *

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Abstract

We describe the recently introduced method of algebraic bosonization of the $(1+1)$ -dimensional Luttinger systems by discussing in detail the specific case of the Calogero-Sutherland model, and mentioning the hard-core Bose gas. We also compare our findings with the exact Bethe Ansatz results.

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It has been known for some time [1, 2] that the one-dimensional gapless fermionic systems solvable by Bethe Ansatz, have leading finite size corrections that are described by the Luttinger model [3]. For this reason, such systems have been called Luttinger systems [2]. More recently [4], it has become clear that the Luttinger model is simply a $c = 1$ conformal field theory. In this talk we will show that also the higher order corrections to the thermodynamic limit of the Luttinger systems have a simple algebraic interpretation [5], namely they are described by an extended conformal field theory based on the $W_{1+\infty} \times \overline{W}_{1+\infty}$ algebra [6, 7], which is an infinite extension of the conformal Virasoro algebra. We illustrate this fact on some specific examples, starting from the Calogero-Sutherland model.

The Calogero-Sutherland [8] model describes N non-relativistic spinless fermions moving on a circle of length L with hamiltonian

$$h = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + g \frac{\pi^2}{L^2} \sum_{j < k} \frac{1}{\sin^2(\pi(x_j - x_k)/L)} \quad ,$$

The second quantized operator corresponding to h is the sum of the kinetic term

$$H_0 = \left(\frac{2\pi}{L} \right)^2 \sum_{n=-\infty}^{\infty} n^2 \psi_n^\dagger \psi_n \quad , \quad (1)$$

and the interaction term [9]

$$H_I = -g \frac{\pi^2}{L^2} \sum_{l,n,m=-\infty}^{\infty} |l| \psi_{m+l}^\dagger \psi_{n-l}^\dagger \psi_n \psi_m \quad , \quad (2)$$

where ψ_n are fermionic oscillators of momentum $k_n = (2\pi/L) n$ ($n \in \mathbf{Z}$ if N is odd and $n \in \mathbf{Z}' \equiv \mathbf{Z} + 1/2$ if N is even), satisfying standard anticommutation relations.

The hamiltonian H_0 describes N free fermions whose ground state is

$$|\Omega\rangle = \psi_{-n_F}^\dagger \dots \psi_{n_F}^\dagger |0\rangle \quad ,$$

with $n_F = (N - 1)/2$. To describe the long-distance properties of the system, it is enough to consider only those oscillators near the Fermi points $\pm n_F$. Thus, the effective theory can be conveniently written in terms of the shifted operators

$$a_r \equiv \psi_{n_F+r+\frac{1}{2}} \quad , \quad b_r \equiv \psi_{-n_F-r-\frac{1}{2}} \quad . \quad (3)$$

The oscillators a_r (b_r) describe small fluctuations of momentum $2\pi r/L$ ($-2\pi r/L$) relative to the right (left) Fermi point. The half-integer index r is allowed to vary only in a finite range between $-\Lambda$ and $+\Lambda$, where the bandwidth cut-off Λ is such that $\Lambda \ll n_F$. Furthermore,

$$a_r |\Omega\rangle = b_r |\Omega\rangle = a_{-r}^\dagger |\Omega\rangle = b_{-r}^\dagger |\Omega\rangle = 0 \quad \text{for } r > 0 \quad . \quad (4)$$

Using these shifted oscillators, the effective hamiltonian corresponding to the kinetic part H_0 reads

$$\begin{aligned} \mathcal{H}_0 &= \left(\frac{2\pi}{L} \right)^2 \sum_{r=-\Lambda}^{\Lambda} \left(n_F + r + \frac{1}{2} \right)^2 \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ &= (2\pi\rho_0)^2 \sum_{r=-\Lambda}^{\Lambda} \left(\frac{1}{4} + \frac{r}{N} + \frac{r^2}{N^2} \right) \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad , \end{aligned} \quad (5)$$

where $\rho_0 = N/L$ is the density, which is held fixed in the thermodynamic limit.

If we keep only the leading $1/N$ -term of \mathcal{H}_0 , *i.e.* if we linearize the dispersion relation around the Fermi points, we can safely let $\Lambda \rightarrow \infty$. Indeed, the spurious states introduced by removing the band-width cut-off, have very high energy, and thus can be neglected in the effective theory. This procedure is the same that was used originally to map the Tomonaga model into the Luttinger model [4]. However, if we want to keep also the higher order terms in the $1/N$ -expansion, we are allowed to extend the sum over r in Eq. (5) up to infinity, provided that at the same time we keep $\Lambda \ll N$ and restrict the Hilbert space to states with relative momentum bounded by Λ [5].

The effective hamiltonian \mathcal{H}_I associated to the interaction term H_I of Eq. (2) can be treated in a similar way; the detailed derivation of \mathcal{H}_I , which requires a suitable regularization procedure and a careful dealing of normal ordering effects in the backward scattering contribution, is presented in Ref. [5]. Here we simply write the complete final result, namely

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I = (2\pi\rho_0)^2 \sum_{k=0}^2 \frac{1}{N^k} \mathcal{H}_{(k)} \quad (6)$$

where

$$\mathcal{H}_{(0)} = \frac{1}{4} (1 + g) \sum_{r \in \mathbf{Z}'} \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \quad , \quad (7)$$

$$\begin{aligned} \mathcal{H}_{(1)} = & \left(1 + \frac{g}{2} \right) \sum_{r \in \mathbf{Z}} r \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ & + \frac{g}{2} \sum_{\ell \in \mathbf{Z}} \sum_{r, s \in \mathbf{Z}'} : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad , \end{aligned} \quad (8)$$

and

$$\begin{aligned} \mathcal{H}_{(2)} = & \sum_{r \in \mathbf{Z}'} \left[r^2 + \frac{g}{4} \left(r^2 - \frac{1}{4} \right) \right] \left(: a_r^\dagger a_r : + : b_r^\dagger b_r : \right) \\ & - \frac{g}{4} \sum_{\ell \in \mathbf{Z}} \sum_{r, s \in \mathbf{Z}'} |\ell| \left(: a_{r-\ell}^\dagger a_r : : a_{s+\ell}^\dagger a_s : + : b_{r+\ell}^\dagger b_r : : b_{s-\ell}^\dagger b_s : \right. \\ & \quad \left. + 2 : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \right) \\ & + \frac{g}{2} \sum_{\ell \in \mathbf{Z}} \sum_{r, s \in \mathbf{Z}'} (r + s - \ell) : a_{r-\ell}^\dagger a_r : : b_{s-\ell}^\dagger b_s : \quad . \end{aligned} \quad (9)$$

Notice that there are no contributions to \mathcal{H} to order $1/N^3$ and higher.

We now show that there is an elegant algebraic structure underlying the effective hamiltonian (6). To see this, we first introduce the fermionic bilinear operators

$$\begin{aligned} V_\ell^0 &= \sum_{r \in \mathbf{Z}'} : a_{r-\ell}^\dagger a_r : \quad , \\ V_\ell^1 &= \sum_{r \in \mathbf{Z}'} \left(r - \frac{\ell}{2} \right) : a_{r-\ell}^\dagger a_r : \quad , \\ V_\ell^2 &= \sum_{r \in \mathbf{Z}'} \left(r^2 - \ell r + \frac{\ell^2}{6} + \frac{1}{12} \right) : a_{r-\ell}^\dagger a_r : \quad , \end{aligned} \quad (10)$$

and \bar{V}_ℓ^0 , \bar{V}_ℓ^1 and \bar{V}_ℓ^2 defined as above with $: a_{r-\ell}^\dagger a_r :$ replaced by $: b_{r-\ell}^\dagger b_r :$. Then, the operators $\mathcal{H}_{(k)}$ of Eqs. (7-9) become

$$\mathcal{H}_{(0)} = \frac{1}{4}(1+g) \left(V_0^0 + \bar{V}_0^0 \right) \quad , \quad (11)$$

$$\mathcal{H}_{(1)} = \left(1 + \frac{g}{2} \right) \left(V_0^1 + \bar{V}_0^1 \right) + \frac{g}{2} \sum_{\ell=-\infty}^{\infty} V_\ell^0 \bar{V}_\ell^0 \quad , \quad (12)$$

$$\begin{aligned} \mathcal{H}_{(2)} = & \left(1 + \frac{g}{4} \right) \left(V_0^2 + \bar{V}_0^2 \right) - \frac{1}{12}(1+g) \left(V_0^0 + \bar{V}_0^0 \right) \\ & - \frac{g}{4} \sum_{\ell=-\infty}^{\infty} |\ell| \left(V_\ell^0 V_{-\ell}^0 + \bar{V}_{-\ell}^0 \bar{V}_\ell^0 + 2 V_\ell^0 \bar{V}_\ell^0 \right) \\ & + \frac{g}{2} \sum_{\ell=-\infty}^{\infty} \left(V_\ell^1 \bar{V}_\ell^0 + V_\ell^0 \bar{V}_\ell^1 \right) \quad . \end{aligned} \quad (13)$$

The operators introduced in Eqs. (10) are the lowest generators of the infinite dimensional $W_{1+\infty}$ algebra [6, 7] whose general form is

$$\left[V_\ell^i, V_m^j \right] = (j\ell - im) V_{\ell+m}^{i+j-1} + q(i, j, \ell, m) V_{\ell+m}^{i+j-3} + \dots + c \delta^{ij} \delta_{\ell+m,0} d(i, \ell) \quad . \quad (14)$$

Here $q(i, j, \ell, m)$ and $d(i, \ell)$ are polynomial structure constants, c is the central charge, and the dots denote a finite number of terms involving the operators $V_{\ell+m}^{i+j-2k}$. In our case $c = 1$, and the commutation relations relevant for our purposes are

$$\left[V_\ell^0, V_m^0 \right] = \ell \delta_{\ell+m,0} \quad , \quad (15)$$

$$\left[V_\ell^1, V_m^0 \right] = -m V_{\ell+m}^0 \quad , \quad (16)$$

$$\left[V_\ell^1, V_m^1 \right] = (\ell - m) V_{\ell+m}^1 + \frac{1}{12} \ell (\ell^2 - 1) \delta_{\ell+m,0} \quad , \quad (17)$$

$$\left[V_\ell^2, V_m^0 \right] = -2m V_{\ell+m}^1 \quad , \quad (18)$$

$$\left[V_\ell^2, V_m^1 \right] = (\ell - 2m) V_{\ell+m}^2 - \frac{1}{6} (m^3 - m) V_{\ell+m}^0 \quad . \quad (19)$$

From Eqs. (15) and (17) we see that the generators V_ℓ^0 satisfy an Abelian Kac-Moody algebra, while the generators V_ℓ^1 close a $c = 1$ Virasoro algebra. The operators \bar{V}_ℓ^i satisfy the same algebra (14) and commute with the V_ℓ^i 's.

The $c = 1$ $W_{1+\infty}$ algebra can be also realized by *bosonic* operators, through a generalized Sugawara construction [7]. In fact, if one introduces the right and left moving modes, α_ℓ and $\bar{\alpha}_\ell$, of a free compactified boson (with the usual commutation relations and canonical normal ordering), one can check that the commutation relations (14) are satisfied by defining V_ℓ^i (we only write the expressions for $i = 0, 1, 2$) as

$$V_\ell^0 = \alpha_\ell \quad , \quad (20)$$

$$V_\ell^1 = \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_n \alpha_{\ell-n} : \quad , \quad (21)$$

$$V_\ell^2 = \frac{1}{3} \sum_{n,m=-\infty}^{\infty} : \alpha_n \alpha_m \alpha_{\ell-n-m} : \quad , \quad (22)$$

and analogously the \bar{V}_ℓ^i in terms of $\bar{\alpha}_\ell$.

The major advantage for choosing the basis of the $W_{1+\infty} \times \bar{W}_{1+\infty}$ operators is that, once the algebraic content of the theory has been established in the free fermionic picture, other bosonic realizations of the *same* algebra can be used, and the free value of the compactification radius of the boson can be chosen to diagonalize the total hamiltonian. This is the reason for calling this procedure *algebraic bosonization* [5].

In the fermionic description it is easy to see that the highest weight states of the $W_{1+\infty} \times \bar{W}_{1+\infty}$ algebra are obtained by adding ΔN particles to the ground state $|\Omega\rangle$, and by moving d particles from the left to the right Fermi point; they are denoted by $|\Delta N, d\rangle_0$. The descendant states,

$$|\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0 = V_{-k_1}^0 \dots V_{-k_r}^0 \bar{V}_{-\bar{k}_1}^0 \dots \bar{V}_{-\bar{k}_s}^0 |\Delta N, d\rangle_0 \quad ,$$

with $k_1 \geq k_2 \geq \dots \geq k_r > 0$, and $\bar{k}_1 \geq \bar{k}_2 \geq \dots \geq \bar{k}_s > 0$, coincide with the particle-hole excitations obtained from $|\Delta N, d\rangle_0$. Using the expressions of V_0^0 and \bar{V}_0^0 given in Eq. (10), one finds that

$$\begin{aligned} V_0^0 |\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0 &= \left(\frac{\Delta N}{2} + d \right) |\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0 \quad , \\ \bar{V}_0^0 |\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0 &= \left(\frac{\Delta N}{2} - d \right) |\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0 \quad . \end{aligned}$$

Thus, the bosonic field built out of the V_ℓ^0 and \bar{V}_ℓ^0 , which describes the density fluctuations of the original free fermions, is compactified on a circle of radius $r_0 = 1$.

Let us now consider the $1/N$ -term of the effective hamiltonian in Eq. (12). Due to the left-right mixing term proportional to g , $\mathcal{H}_{(1)}$ is not diagonal on the descendant states $|\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0$. However, it can be diagonalized [3] using the Sugawara construction: replacing $(V_0^1 + \bar{V}_0^1)$ with the expression given by Eq. (21), we obtain a quadratic form in V_ℓ^0 and \bar{V}_ℓ^0 , which can be now diagonalized by means of the following Bogoliubov transformation

$$\begin{aligned} W_\ell^0 &= V_\ell^0 \cosh \beta + \bar{V}_{-\ell}^0 \sinh \beta \quad , \\ \bar{W}_\ell^0 &= V_{-\ell}^0 \sinh \beta + \bar{V}_\ell^0 \cosh \beta \end{aligned} \quad (23)$$

for all ℓ , with $\tanh 2\beta = g/(2+g)$.

The operators W_ℓ^0 and \bar{W}_ℓ^0 satisfy the same abelian Kac-Moody algebra with central charge $c = 1$ as the original operators V_ℓ^0 and \bar{V}_ℓ^0 (cf. Eq. (15)). Then, by means of the generalized Sugawara construction, we can define a new realization of the $W_{1+\infty}$ algebra whose generators W_ℓ^i and \bar{W}_ℓ^i are forms of degree $(i+1)$ in W_ℓ^0 and \bar{W}_ℓ^0 respectively. Consequently, the effective hamiltonian, up to order $1/N$, becomes [5]

$$\begin{aligned} \mathcal{H}_{(1/N)} &\equiv (2\pi\rho_0)^2 \left(\mathcal{H}_{(0)} + \frac{1}{N} \mathcal{H}_{(1)} \right) \\ &= (2\pi\rho_0\sqrt{\lambda})^2 \left[\left(\frac{\sqrt{\lambda}}{4} W_0^0 + \frac{1}{N} W_0^1 \right) + (W \leftrightarrow \bar{W}) \right] \quad , \end{aligned} \quad (24)$$

where

$$\lambda \equiv \exp(2\beta) = \sqrt{1+g} \quad . \quad (25)$$

Notice that $\mathcal{H}_{(1/N)}$ is not diagonal on $|\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_0$, because the highest weight states of the new algebra do not coincide with the vectors $|\Delta N, d\rangle_0$, as is clear from Eq. (23). However, the Bogoliubov transformation does not mix states belonging to different Verma moduli. This implies that the new highest weight vectors, $|\Delta N; d\rangle_W$, are still characterized by the numbers ΔN and d with the same meaning as before, but their charges are different. More precisely

$$\begin{aligned} W_0^0 |\Delta N; d\rangle_W &= \left(\sqrt{\lambda} \frac{\Delta N}{2} + \frac{d}{\sqrt{\lambda}} \right) |\Delta N; d\rangle_W \\ \overline{W}_0^0 |\Delta N; d\rangle_W &= \left(\sqrt{\lambda} \frac{\Delta N}{2} - \frac{d}{\sqrt{\lambda}} \right) |\Delta N; d\rangle_W \quad . \end{aligned} \quad (26)$$

From the last two equations we deduce that W_ℓ^0 and \overline{W}_ℓ^0 are still the modes of a compactified bosonic field, but on a circle of radius $r = 1/\sqrt{\lambda} = \exp(-\beta)$. This new field describes the density fluctuations of the *interacting* fermions.

The highest weight states $|\Delta N, d\rangle_W$ together with their descendants, which we denote by $|\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_W$, form a *new* bosonic basis for our theory that has no simple expression in terms of the original free fermionic degrees of freedom. The main property of this new basis is that it diagonalizes the effective hamiltonian up to order $1/N$. In fact, using Eqs. (24) and (26), it is easy to check that

$$\mathcal{H}_{(1/N)} |\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_W = \mathcal{E}_{(1/N)} |\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_W$$

where [10]

$$\mathcal{E}_{(1/N)} = \left(2\pi\rho_0\sqrt{\lambda} \right)^2 \left[\frac{\lambda}{4} \Delta N + \frac{1}{N} \left(\lambda \frac{(\Delta N)^2}{4} + \frac{d^2}{\lambda} + k + \bar{k} \right) \right] \quad , \quad (27)$$

with $k = \sum_i k_i$ and $\bar{k} = \sum_j \bar{k}_j$. These eigenvalues are clearly degenerate when $k \geq 2$ or $\bar{k} \geq 2$.

Since the effective hamiltonian \mathcal{H} has been derived within a perturbative approach, an expansion in the coupling constant g should be understood in all previous formulae. However, if we limit our analysis to the $1/N$ -terms, nothing prevents us from improving our results by extending them to all orders in g . Indeed, the Bogoliubov transformation (23) diagonalizes the hamiltonian $\mathcal{H}_{(1)}$ *exactly*, and the resulting expression depends on the coupling constant only through λ , which contains all powers of g ! This improvement is a well-known result in the Luttinger model [3, 4], but we would like to stress that in our case it can be done only if we disregard the $O(1/N^2)$ -terms of the hamiltonian, because the Bogoliubov transformation (23) does not diagonalize $\mathcal{H}_{(2)}$.

To investigate this issue, let us analyze the $1/N^2$ -term of the effective hamiltonian Eq. (13). Using the generalized Sugawara construction, Eqs. (20–22), we first rewrite $\mathcal{H}_{(2)}$ as a cubic form in V_ℓ^0 and \overline{V}_ℓ^0 , and then perform the Bogoliubov transformation (23) to

re-express it in terms of the W_ℓ^i and \overline{W}_ℓ^i generators. A straightforward calculation leads to $\mathcal{H}_{(2)} = \mathcal{H}'_{(2)} + \mathcal{H}''_{(2)}$, where

$$\mathcal{H}'_{(2)} = \sqrt{\lambda} \left(W_0^2 + \overline{W}_0^2 \right) - \frac{\sqrt{\lambda^3}}{12} \left(W_0^0 + \overline{W}_0^0 \right) - \frac{g}{2\lambda} \sum_{\ell=1}^{\infty} \ell \left(W_{-\ell}^0 W_\ell^0 + \overline{W}_{-\ell}^0 \overline{W}_\ell^0 \right) \quad , \quad (28)$$

and

$$\mathcal{H}''_{(2)} = -\frac{g}{2\lambda} \sum_{\ell=1}^{\infty} \ell \left(W_\ell^0 \overline{W}_\ell^0 + W_{-\ell}^0 \overline{W}_{-\ell}^0 \right) \quad . \quad (29)$$

Neither $\mathcal{H}'_{(2)}$ nor $\mathcal{H}''_{(2)}$ are diagonal on the states $|\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_W$ considered so far. In fact, these states are not in general eigenstates of $(W_0^2 + \overline{W}_0^2)$, and hence cannot be eigenstates of $\mathcal{H}'_{(2)}$; moreover, since they have definite values of k and \bar{k} , they cannot be eigenstates of $\mathcal{H}''_{(2)}$ either, because this operator mixes the left and right sectors.

It is not difficult, however, to overcome these problems. Since $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$ commute with each other, it is always possible to find suitable combinations of the states $|\Delta N, d; \{k_i\}, \{\bar{k}_j\}\rangle_W$ with fixed k and \bar{k} that diagonalize simultaneously $\mathcal{H}'_{(2)}$ and $\mathcal{H}_{(1/N)}$, therefore lifting the degeneracy of the spectrum present to order $1/N$. The term $\mathcal{H}''_{(2)}$, instead, has to be treated perturbatively, but only to first order in g . In fact to higher orders, the spurious states introduced extending to infinity the sum in Eq. (5) would contribute as intermediate states. These contributions, however, would be meaningless because the hamiltonian to order $O(1/N^2)$ is not even bounded below. From Eq. (29) it is easy to check that $\mathcal{H}''_{(2)}$ has vanishing expectation value on any state that is simultaneously eigenstate of $\mathcal{H}_{(1/N)}$ and $\mathcal{H}'_{(2)}$. Thus, according to (non-degenerate) perturbation theory, $\mathcal{H}''_{(2)}$ has no effect on the energy spectrum to first order in g .

In view of these considerations, we neglect $\mathcal{H}''_{(2)}$ and regard as the effective hamiltonian the following operator

$$\begin{aligned} \mathcal{H} &\equiv \mathcal{H}_{(1/N)} + (2\pi\rho_0)^2 \frac{1}{N^2} \mathcal{H}'_{(2)} \\ &= \left(2\pi\rho_0\sqrt{\lambda} \right)^2 \left\{ \left[\frac{\sqrt{\lambda}}{4} W_0^0 + \frac{1}{N} W_0^1 + \frac{1}{N^2} \left(\frac{1}{\sqrt{\lambda}} W_0^2 - \frac{\sqrt{\lambda}}{12} W_0^0 \right. \right. \right. \\ &\quad \left. \left. \left. - \frac{g}{2\lambda^2} \sum_{\ell=1}^{\infty} \ell W_{-\ell}^0 W_\ell^0 \right) \right] + (W \leftrightarrow \overline{W}) \right\} \quad . \end{aligned} \quad (30)$$

Obviously, to be consistent with our perturbative approach, we should keep in the r.h.s. of Eq. (30) only the terms that are linear in g .

We now compare the eigenvalues of \mathcal{H} to the exact low-energy spectrum of the Calogero-Sutherland model obtained from the Bethe Ansatz method [10]. Any low-energy solution of the Bethe Ansatz equations is labeled by a set of integer numbers

$$I_j = \frac{2j-1-N'}{2} + d - \bar{n}_j + n_{N'-j+1} \quad (31)$$

where $N' = N + \Delta N$, and $j = 1, \dots, N'$. The integers n_j are ordered according to $n_1 \geq n_2 \geq \dots \geq 0$ and are different from zero only if $j < \Lambda \ll N$ (and analogously for \bar{n}_j).

By generalizing to order $1/N^2$ the procedure presented in Ref. [10], we have derived [5] the exact energy of the excitation described by the numbers (31):

$$\begin{aligned} \tilde{\mathcal{E}} = & \left(2\pi\rho_0\sqrt{\xi}\right)^2 \left\{ \left[\frac{\sqrt{\xi}}{4} Q + \frac{1}{N} \left(\frac{1}{2} Q^2 + \sum_j n_j \right) \right. \right. \\ & + \frac{1}{N^2} \left(\frac{1}{3\sqrt{\xi}} Q^3 - \frac{\sqrt{\xi}}{12} Q + \frac{2\sum_j n_j}{\sqrt{\xi}} Q + \frac{\sum_j n_j^2}{\xi} - \sum_j (2j-1) n_j \right) \\ & \left. \left. + \left(Q \leftrightarrow \overline{Q}, \{n_j\} \leftrightarrow \{\overline{n}_j\} \right) \right] \right\} , \end{aligned} \quad (32)$$

where

$$\xi = \frac{1 + \sqrt{1+2g}}{2} , \quad (33)$$

and

$$Q = \sqrt{\xi} \frac{\Delta N}{2} + \frac{d}{\sqrt{\xi}} , \quad \overline{Q} = \sqrt{\xi} \frac{\Delta N}{2} - \frac{d}{\sqrt{\xi}} . \quad (34)$$

Of course, being an exact result, Eq. (32) holds to all orders in g . Comparing Eqs. (25) and (33), we see that

$$\xi = \lambda + O(g^2) . \quad (35)$$

Comparing Eq. (32) with Eq. (27), we realize that, at least to order $1/N$, the exact results can be obtained from the perturbative ones simply by changing λ into ξ . Moreover, in Ref. [5] we have checked on several explicit examples that the eigenvalues of \mathcal{H} with ξ in place of λ coincide with the exact energy of the low-lying excitations given by Eq. (32). Thus, we are led to conjecture that the *exact* effective hamiltonian of the Calogero-Sutherland model is given by Eq. (30) with ξ in place of λ . We may consider this operator as a non-perturbative improvement of \mathcal{H} which was derived in perturbation theory.

We conclude by mentioning that our method of algebraic bosonization can be applied in principle to any gapless fermionic hamiltonian consisting of a bilinear kinetic term and an arbitrary four-fermion interaction. No special requirements on the form of the dispersion relation and the potential are needed. In particular, it is not necessary for the system to be integrable. In Ref. [5] we have also discussed the algebraic bosonization of the Heisenberg model, by mapping it into a theory of fermions on a lattice by means of a Jordan-Wigner transformation.

By comparison with the Bethe Ansatz solution it can be shown [11] that also the one-dimensional Bose gas, with hamiltonian

$$H = \int_0^L dx \left[\partial_x \phi^\dagger(x) \partial_x \phi(x) + 2\kappa \phi^\dagger(x) \phi^\dagger(x) \phi(x) \phi(x) \right] , \quad (36)$$

with $\kappa > 0$, has a spectrum of low-energy excitations that can be described by the representation theory of the $W_{1+\infty} \times \overline{W}_{1+\infty}$ algebra. In particular, to first order in a $1/\kappa$ -expansion, the effective hamiltonian corresponding to Eq. (36) turns out to be

$$\mathcal{H} = (2\pi\rho_0)^2 \left\{ \frac{1}{4} \left(1 - \frac{5}{3}g \right) W_0^0 + \frac{1}{N} (1 - 2g) W_0^1 \right.$$

$$\begin{aligned}
& + \frac{1}{N^2} \left[(1 - 2g) W_0^2 - \frac{1 - 3g}{12} W_0^0 - 2g W_0^0 \overline{W}_0^0 \right] \\
& - \frac{2g}{N^3} \left[(W_0^2 + \overline{W}_0^2) W_0^0 - \frac{1}{2} (W_0^1 - \overline{W}_0^1)^2 - \frac{1}{24} (W_0^0 + \overline{W}_0^0)^2 \right] + (W \leftrightarrow \overline{W}) \Big\}
\end{aligned} \tag{37}$$

where $\rho_0 = N/L$ and $g = 2\rho_0/\kappa$. Turning to the fermionic realization of the $W_{1+\infty} \times \overline{W}_{1+\infty}$ algebra, one can calculate the first order correction to the gas of free fermions, which is equivalent to the gas of one-dimensional boson ($\kappa = \infty$), and find the following hamiltonian

$$H = - \int_0^L dx \, \psi^\dagger(x) \partial_x^2 \psi(x) + \frac{2}{\kappa L} \int_0^L dx \int_0^L dy \, \psi^\dagger(x) \psi^\dagger(y) (\partial_x - \partial_y)^2 \psi(x) \psi(y) \quad .$$

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